Pergamon

Int. J. Solids Structures Vol. 35, Nos 26–27, pp. 3519–3537, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00212–6

MORE ON STRESS INVARIANCE CONDITIONS FOR THE TRACTION BOUNDARY VALUE PROBLEM OF PLANE LINEAR ELASTICITY

Q.-C. HE

Laboratory of Applied Mechanics, Department of Mechanical Engineering, Swiss Federal Institute of Technology, CH-1015 Lausanne, Switzerland

(*Received* 15 *January* 1997; *in revised form* 22 *July* 1997)

Abstract—Cherchaev *et al.* (1992) gave a sufficient condition for a change in the elastic tensor field to preserve the stress state of a plane linearly elastic solid subject to prescribed traction forces on its boundary. This condition turned out to have a number of fruitful applications in the mechanics of composite materials and was later extended by Dundurs and Markenscoff (1993). The present work addresses and answers the question whether there is a yet more general condition. We show that: (i) if the change in the elastic tensor field is required to be hyperelastic, the extended condition of Dundurs and Markenscoff is not only sufficient but also necessary; (ii) if the change in the elastic tensor field is relaxed so as to be elastic, a more general necessary and sufficient condition exists. In proving these two conclusions, several orthogonal decompositions are constructed for third- and fourth-order tensors presenting index permutation symmetries. Such decompositions are probably also useful for solving other problems in mechanics.

1. INTRODUCTION

Consider a plane solid $\mathscr{B}$ occupying, in its reference configuration, an open, bounded, simply-connected domain Ω of a two-dimensional (2-D) Euclidean point space R^2, with $\partial\Omega$ as the boundary (Fig. 1). Under the action of some prescribed tractions $\mathbf{t}$ on $\partial\Omega$, $\mathscr{B}$ undergoes small deformations and the material M constituting $\mathscr{B}$ behaves as a linearly elastic one so as to be characterized by a 2-D elastic tensor field $\mathbb{K}(\mathbf{x})$ over $\bar{\Omega} := \Omega \cup \partial\Omega$. In such a context, Cherkaev *et al.* (1992) proved that substituting M by another linearly elastic material M' does not alter the stress tensor field $\mathbf{T}(\mathbf{x})$ over $\bar{\Omega}$ if the elastic tensor field $\mathbb{K}'(\mathbf{x})$ of M' is related to $\mathbb{K}(\mathbf{x})$ by

$$\mathbb{K}'(\mathbf{x}) = \mathbb{K}(\mathbf{x}) + c\mathbb{R}, \quad \mathbb{R} := \mathbf{1} \otimes \mathbf{1} - \mathbb{I}. \tag{1.1}$$

Here, c denotes a constant scalar, $\otimes$ the usual tensor product operation, $\mathbf{1}$ the identity tensor on a 2-D Euclidean space $\mathscr{V}$, and $\mathbb{I}$ the identity on the space Sym of 2nd-order symmetric tensors on $\mathscr{V}$. The result of Cherkaev *et al.* is of both theoretical and practical importance and is now referred to as the CLM theorem. Since its publication, it has inspired

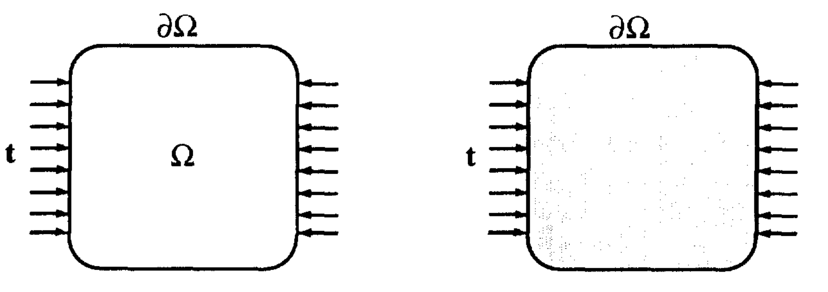

Fig. 1. The linearly elastic material constituting $\mathscr{B}$ is characterized by $\mathbb{K}(\mathbf{x})$ in (a) and by $\mathbb{K}'(\mathbf{x})$ in (a').

a good few investigators (Thorpe and Jasiuk, 1992; Christensen, 1993; Dundurs and Markenscoff, 1993; Jun and Jasiuk, 1993; Jasiuk *et al.*, 1994; Moran and Gosz, 1994; Chen, 1995; Milton and Movchan, 1995; Ostoja-Starzewski and Jasiuk, 1995; Zheng and Hwang, 1996, 1997). Of them, Dundurs and Markenscoff (1993) generalized the CLM theorem in two directions, firstly by allowing $\mathscr{B}$ to be multiply-connected and made of a non smoothly inhomogenous material, and secondly by extending the relation between $\mathbb{K}'(\mathbf{x})$ and $\mathbb{K}(\mathbf{x})$ from the constant one (1.1) to the following affine one:

$$\mathbb{K}'(\mathbf{x}) = \mathbb{K}(\mathbf{x}) + (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R}, \tag{1.2}$$

where $\mathbf{b}\cdot\mathbf{x}$ stands for the scalar product of $\mathbf{x}\in\bar{\Omega}$ by a constant vector $\mathbf{b}\in\mathscr{V}$.

The present work was initiated by the following question:

Is there a relation more general than (1.2)?

To the best of the author's knowledge this question is open. Clearly, the answer to it can have only two possibilities: (i) there is no relation more general than (1.2); (ii) there is effectively a relation more general than (1.2). The first possible answer means that (1.2) is not only sufficient but also necessary. If this answer were proved to be correct, it would complete the conclusion of Dundurs and Markenscoff while clarifying the necessity of (1.2). The second possible answer amounts to finding a relation which, ideally not only sufficient but also necessary, includes (1.2) as a particular one. If such a relation were proved to exist, relation (1.2) would further be extended. So, in any case the answer to our question is constructive.

To make our question precise, let us describe, with the help of Fig. 1, the problem originally set by Cherkaev *et al.* (1992) in an extended and complete way. The data for the traction boundary value problem of plane linear elasticity are $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ in the case (a), and $(\bar{\Omega}, \mathbb{K}', \mathbf{t})$ in the case (a′). Without loss of generality, $\mathbb{K}'$ can be considered as being related to $\mathbb{K}$ by

$$\mathbb{K}'(\mathbf{x}) = \mathbb{K}(\mathbf{x}) + \mathbb{D}(\mathbf{x}), \tag{1.3}$$

where $\mathbb{D}(\mathbf{x})$ is the difference between the fields $\mathbb{K}'(\mathbf{x})$ and $\mathbb{K}(\mathbf{x})$. As the domain $\bar{\Omega}$ and tractions $\mathbf{t}$ acting on $\partial\Omega$ are the same in both cases, we are naturally led to ask what are the functions for $\mathbb{D}(\mathbf{x}) \neq \mathbb{0}$, such that the stress field $\mathbf{T}'(\mathbf{x})$ in the case (a′) is identical to the stress field $\mathbf{T}(\mathbf{x})$ in the case (a). If one of these two fields, say $\mathbf{T}(\mathbf{x})$, is taken as the reference stress field, it is clear that, among the functions thus sought for $\mathbb{D}(\mathbf{x})$, certain may depend on the specific data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$, since $\mathbf{T}(\mathbf{x})$ does. However, such functions present no general interests because of their data dependence. Thus, our question becomes the problem of finding out all those functions for $\mathbb{D}(\mathbf{x}) \neq \mathbb{0}$, such that they are independent of any data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ and that the stress states in the cases (a) and (a′) are identical. This problem is solved in the present work. We show that:

- if $\mathbb{D}$ is required to be hyperelastic, i.e. the Cartesian matrix components D_{ijmn} of $\mathbb{D}$ have the index permutation symmetries $D_{ijmn} = D_{jimn} = D_{mnij}$, then the expression $\mathbb{D}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R}$ is general, so that (1.2) is not only sufficient but also necessary;
- if $\mathbb{D}(\mathbf{x})$ is relaxed so as to be elastic, i.e. $D_{ijmn} = D_{jimn} = D_{ijnm}$, then it has a more general necessary and sufficient expression: $\mathbb{D}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R} + \mathbb{V}(\mathbf{x})$, where $\mathbb{V}(\mathbf{x})$ is a quadratic polynomial function of $\mathbf{x}$ and specified by eqn (3.35b).

The outline of this paper is as follows. In Section 2, we first present the notation used throughout the paper, and then derive several orthogonal unique decompositions for third- and fourth-order tensors with index permutation symmetries, which are needed for proving the aforementioned two conclusions. Most of these decompositions are new to the knowledge of the author. In Section 3, the foregoing problem is carefully formulated so that the condition governing $\mathbb{D}(\mathbf{x})$ is established. In solving the problem, we proceed in two steps, successively considering $\mathbb{D}(\mathbf{x})$ to be hyperelastic and elastic. In both cases, the strategy adopted consists in using the solutions of some particular traction boundary value problems of plane linear elasticity to deduce the necessary conditions for $\mathbb{D}(\mathbf{x})$, and checking whether the last necessary conditions obtained for $\mathbb{D}(\mathbf{x})$ are also sufficient. This strategy runs counter to that employed by Cherchaev *et al.* (1992), Dundurs and Markenscoff (1993) and others, who proceeded from getting the sufficient conditions for $\mathbb{D}(\mathbf{x})$. Our strategy is technically applicable owing to the orthogonal decompositions constructed in Section 2. The paper ends with a few concluding comments given in Section 4.

2. PRELIMINARIES

This section is essentially concerned with some algebraic properties of 2-D third- and fourth-order tensors exhibiting index permutation symmetries. As will be seen in Section 3, these properties play a crucial role in solving the problem of this paper. However, as their derivation is independent of that problem, they have wider scope.

2.1. *Notation. Right-angle rotation. The Schwarz theorem*

The notation presented here is similar to that adopted in He and Curnier (1995) and He (1997). Throughout the paper, coordinate-free notation will be employed as much as possible. Bold-face (outline) Latin minuscule and majuscule letters will be used to denote vectors (3rd-order) and 2nd-order (4th-order) tensors, respectively. Lin, $\mathbb{l}$in and $\mathbb{L}$in will stand for the respective spaces of 2nd-, 3rd- and 4th-order tensors over a 2-D Euclidean space $\mathscr{V}$. The inner product of any vector space will be designated by a bold-face dot $\cdot$; for example, $\mathbf{a}\cdot\mathbf{b}$ for $\mathbf{a}, \mathbf{b}\in\mathscr{V}$, and $\mathbb{A}\cdot\mathbb{B}$ for $\mathbb{A}, \mathbb{B}\in\mathbb{L}$in.

Given $\mathbf{b}\in\mathscr{V}$ and $\mathbf{A}, \mathbf{B}\in$ Lin, we define $\mathbf{A}\underline{\otimes}\mathbf{b}$, $\mathbf{A}\underline{\otimes}\mathbf{B}$ and $\mathbf{A}\overline{\otimes}\mathbf{B}$ by setting, for any $\mathbf{u}$, $\mathbf{v}\in\mathscr{V}$:

$$(\mathbf{A}\underline{\otimes}\mathbf{b})\mathbf{u} := (\mathbf{A}\mathbf{u})\otimes\mathbf{b}, \quad (\mathbf{A}\underline{\otimes}\mathbf{B})(\mathbf{u}\otimes\mathbf{v}) := (\mathbf{A}\mathbf{u})\otimes(\mathbf{B}\mathbf{v}), \tag{2.1a}$$

$$(\mathbf{A}\overline{\otimes}\mathbf{B})(\mathbf{u}\otimes\mathbf{v}) := (\mathbf{A}\mathbf{v})\otimes(\mathbf{B}\mathbf{u}). \tag{2.1b}$$

The products $\mathbf{A}\underline{\otimes}\mathbf{B}$ and $\mathbf{A}\overline{\otimes}\mathbf{B}$ are of Kronecker type. With the help of the identities $(\mathbf{A}\mathbf{u})\otimes(\mathbf{B}\mathbf{v}) = \mathbf{A}(\mathbf{u}\otimes\mathbf{v})\mathbf{B}^T$ and $(\mathbf{u}\otimes\mathbf{v})^T = \mathbf{v}\otimes\mathbf{u}$, it can be verified that, for each $\mathbf{X}\in$ Lin,

$$(\mathbf{A}\underline{\otimes}\mathbf{B})\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{B}^T, \quad (\mathbf{A}\overline{\otimes}\mathbf{B})\mathbf{X} = \mathbf{A}\mathbf{X}^T\mathbf{B}^T. \tag{2.2}$$

Thus, the identity $\mathbb{1}$ and transposition $\mathbb{T}$ on **Lin** as well as the identity $\mathbb{I}$ on Sym have their coordinate-free expressions:

$$\mathbb{1} = \mathbf{1}\underline{\otimes}\mathbf{1}, \quad \mathbb{T} = \mathbf{1}\overline{\otimes}\mathbf{1}, \quad \mathbb{I} = \tfrac{1}{2}(\mathbf{1}\underline{\otimes}\mathbf{1}+\mathbf{1}\overline{\otimes}\mathbf{1}). \tag{2.3}$$

In addition to (2.2), the following identities will be useful:

$$(\mathbf{a}\otimes\mathbf{b})\underline{\otimes}(\mathbf{c}\otimes\mathbf{d}) = \mathbf{a}\otimes\mathbf{c}\otimes\mathbf{b}\otimes\mathbf{d}, \quad (\mathbf{a}\otimes\mathbf{b})\overline{\otimes}(\mathbf{c}\otimes\mathbf{d}) = \mathbf{a}\otimes\mathbf{c}\otimes\mathbf{d}\otimes\mathbf{b}, \tag{2.4a}$$

$$(\mathbf{A}\underline{\otimes}\mathbf{B})(\mathbf{C}\underline{\otimes}\mathbf{D}) = (\mathbf{A}\mathbf{C})\underline{\otimes}(\mathbf{B}\mathbf{D}), \quad (\mathbf{A}\overline{\otimes}\mathbf{B})(\mathbf{C}\overline{\otimes}\mathbf{D}) = (\mathbf{A}\mathbf{D})\underline{\otimes}(\mathbf{B}\mathbf{C}), \tag{2.4b}$$

$$(\mathbf{A}\underline{\otimes}\mathbf{B})(\mathbf{C}\overline{\otimes}\mathbf{D}) = (\mathbf{A}\mathbf{C})\overline{\otimes}(\mathbf{B}\mathbf{D}), \quad (\mathbf{A}\overline{\otimes}\mathbf{B})(\mathbf{C}\underline{\otimes}\mathbf{D}) = (\mathbf{A}\mathbf{D})\overline{\otimes}(\mathbf{B}\mathbf{C}). \tag{2.4c}$$

If $\mathbf{Q}\in$ Lin is orthogonal, i.e. $\mathbf{Q}\mathbf{u}\cdot\mathbf{Q}\mathbf{y} = \mathbf{u}\cdot\mathbf{v}$ for all $\mathbf{u}, \mathbf{v}\in\mathscr{V}$, then $\mathbb{Q} := (\mathbf{Q}\underline{\otimes}\mathbf{Q}+\mathbf{Q}\overline{\otimes}\mathbf{Q})/2$ is a rotation on the subspace $\text{Sym} := \{\mathbf{S}\in\text{Lin}\,|\,\mathbf{S}\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{S}\mathbf{v},\ \forall\mathbf{u},\ \mathbf{v}\in\mathscr{V}\}$, since $\det(\mathbb{Q}) = 1$ and $\mathbb{Q}\mathbf{U}\cdot\mathbb{Q}\mathbf{V} = \mathbf{U}\cdot\mathbf{V}$ for all $\mathbf{U}, \mathbf{V}\in$ Sym. Letting $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis for $\mathscr{V}$, the right-angle rotation $\mathbf{R}$ from $\mathbf{e}_2$ to $\mathbf{e}_1$ is represented by

$$\mathbf{R} = \mathbf{e}_1\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_1. \tag{2.5}$$

This tensor is equally the 2-D alternator, because its Cartesian components have the property that $R_{11} = R_{22} = 0$ and $R_{12} = -R_{21} = 1$. In view of (2.4a) and the fact that $\mathbf{1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2$, we can relate $\mathbb{R}$ to $\mathbf{R}$ by

$$\mathbb{R} = \mathbf{1} \otimes \mathbf{1} - \mathbb{I} = \tfrac{1}{2}(\mathbf{R} \underline{\otimes} \mathbf{R} + \mathbf{R} \overline{\otimes} \mathbf{R}). \tag{2.6}$$

This means that $\mathbb{R}$ is the right-angle rotation on Sym. Cherkaev *et al.* (1992) seem to be the first to give such an interpretation to $\mathbb{R}$. Applying (2.4b) and (2.4c) to $\mathbb{R}$ with its domain and range restricted to Sym yields another remarkable property of $\mathbb{R}$ (He, 1997):

$$\mathbb{R} = \mathbb{R}^{-1} = \mathbb{R}^T, \tag{2.7}$$

where $\mathbb{R}^{-1}$ and $\mathbb{R}^T$ are defined by $\mathbb{R}^{-1}\mathbb{R} = \mathbb{R}\mathbb{R}^{-1} = \mathbb{I}$ and $\mathbb{R}^T\mathbf{U}\cdot\mathbf{V} = \mathbf{U}\cdot\mathbb{R}\mathbf{V}$ for every $\mathbf{U}$, $\mathbf{V} \in \mathrm{Sym}$.

The notion of symmetric tensors is familiar when the order n of tensors is equal to two. To extent it to the case of $n > 2$, we write the usual definition of the symmetry of $\mathbf{S} \in \mathrm{Sym}$, i.e. $\mathbf{S}\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{S}\mathbf{u}_2$ for all $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathscr{V}$, in the equivalent form: $\mathbf{S} \cdot (\mathbf{u}_2 \otimes \mathbf{u}_1) = \mathbf{S} \cdot (\mathbf{u}_1 \otimes \mathbf{u}_2)$. In a similar manner, a tensor $\mathbf{S}$ of order $n \geqslant 2$ is said to be symmetric if

$$\mathbf{S} \cdot (\mathbf{u}_{\sigma(1)} \otimes \mathbf{u}_{\sigma(2)} \otimes \cdots \otimes \mathbf{u}_{\sigma(n)}) = \mathbf{S} \cdot (\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \ldots \otimes \mathbf{u}_n) \tag{2.8}$$

for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathscr{V}$ and for all permutation σ of the set $\{1, 2, \ldots, n\}$. In matrix notation, (2.8) means that a Cartesian component $S_{i_1 i_2 \ldots i_n}$ of $\mathbf{S}$ is invariant under all permutation of indices. This amounts to writing

$$S_{i_1 i_2 \ldots i_n} = \frac{1}{n!} S_{(i_1 i_2 \ldots i_n)}, \tag{2.9}$$

where $S_{(i_1 i_2 \ldots i_n)}$ denotes the sum of the terms obtained by permuting the indices of $S_{i_1 i_2 \ldots i_n}$ in all $n!$ possible ways. In practice, (2.9) constitutes a computationally convenient criterion for verifying whether a given tensor of order $n \geqslant 2$ is symmetric.

In close connection with the above notion of symmetric tensors is the classical Schwarz theorem, which will play a key role in Section 3. Let φ, $\mathbf{g}$ and $\mathbf{G}$ be sufficiently smooth scalar, vector and 2nd-order tensor value functions on $\bar{\Omega} \subset \mathrm{R}^2$. For their derivatives we shall use the notations:

$$\nabla\varphi = \varphi_{,i}\mathbf{e}_i, \quad (\nabla \otimes \nabla)\varphi = \varphi_{,ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.10a}$$

$$(\nabla \otimes \nabla \otimes \nabla)\varphi = \varphi_{,ijm}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m, \quad (\nabla \otimes \nabla \otimes \nabla \otimes \nabla)\varphi = \varphi_{,ijmn}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \tag{2.10b}$$

$$\nabla \cdot \mathbf{g} = g_{i,i}, \quad \mathbf{G}\nabla = G_{ij,j}\mathbf{e}_i, \quad (\nabla \otimes \nabla) \cdot \mathbf{G} = G_{ij,ij}, \tag{2.10c}$$

where the indices following a comma indicate partial differentiations. The Schwarz theorem asserts that the tensors $(\nabla \otimes \nabla)\varphi$, $(\nabla \otimes \nabla \otimes \nabla)\varphi$ and $(\nabla \otimes \nabla \otimes \nabla \otimes \nabla)\varphi$ are all symmetric, i.e., that their Cartesian matrix components have the property that

$$\varphi_{,ij} = \tfrac{1}{2!}\varphi_{,(ij)}, \quad \varphi_{,ijm} = \tfrac{1}{3!}\varphi_{,(ijm)}, \quad \varphi_{,ijmn} = \tfrac{1}{4!}\varphi_{,(ijmn)}, \tag{2.11}$$

provided φ is four times continuously differentiable or of class $C^{(4)}$ for short.

2.2. *Decomposition of 3rd- and 4th-order tensors exhibiting index permutation symmetries*

It has been shown (Backus, 1970; Spencer, 1970; He, 1997) that the classical orthogonal decomposition of a 2nd-order tensor $\mathbf{L} \in \mathrm{Lin}$ into a unique symmetric tensor

$\mathbf{S} = (\mathbf{L}+\mathbf{L}^T)/2$ and a unique antisymmetric tensor $\mathbf{A} = (\mathbf{L}-\mathbf{L}^T)/2$ can be extended to a tensor $\mathbf{L}$ of order $n > 2$. In the extended sense, a tensor $\mathbf{A}$ of order $n \geqslant 2$ is said to be antisymmetric if its matrix components $A_{i_1 i_2 \ldots i_n}$ verify the condition that

$$A_{(i_1 i_2 \ldots i_n)} = 0. \tag{2.12}$$

With any given element $\mathbf{L}$ of the space $\mathscr{L}$ of nth-order tensors ($n \geqslant 2$), we can associate two tensors, $\mathbf{S}$ and $\mathbf{A}$, whose matrix components are calculated in terms of those of $\mathbf{L}$ by

$$S_{i_1 i_2 \ldots i_n} := \frac{1}{n!} L_{(i_1 i_2 \ldots i_n)}, \quad A_{i_1 i_2 \ldots i_n} := L_{i_1 i_2 \ldots i_n} - S_{i_1 i_2 \ldots i_n}. \tag{2.13}$$

By this construction, $\mathbf{S}$ is symmetric and $\mathbf{A}$ is antisymmetric, for they satisfy (2.9) and (2.12), respectively. In addition, $\mathbf{S}$ and $\mathbf{A}$ are orthogonal:

$$\mathbf{S} \cdot \mathbf{A} = S_{i_1 i_2 \ldots i_n} A_{i_1 i_2 \ldots i_n} = 0. \tag{2.14}$$

This stems from both the symmetry of $\mathbf{S}$ and the antisymmetry of $\mathbf{A}$; indeed, owing to (2.9) and (2.12), $S_{i_1 i_2 \ldots i_n} A_{i_1 i_2 \ldots i_n} = S_{i_1 i_2 \ldots i_n} A_{(i_1 i_2 \ldots i_n)}/n! = 0$. A summary account of $(2.13)_2$ and (2.14) consists in writing

$$\mathscr{L} = \mathscr{S} \oplus \mathscr{A}, \tag{2.15}$$

where $\oplus$ represents the orthogonal direct sum of the respective subspaces, $\mathscr{S}$ and $\mathscr{A}$, formed by all nth-order symmetric and antisymmetric tensors over $\mathscr{V}$. Uniqueness of decomposition (2.15) can be proved in the same way as that used for the well-known one $\mathbf{L} = \mathbf{S}+\mathbf{A}$.

Before applying (2.13) and (2.15) to 3rd- and 4th-order tensors presenting index permutation symmetries, let us define what we mean by an index permutation symmetry of $\mathbf{L}$. Such a symmetry will refer to a non-identity permutation σ of the set $\{1, 2, \ldots n\}$ verifying the condition that

$$\mathbf{L} \cdot (\mathbf{u}_{\sigma(1)} \otimes \mathbf{u}_{\sigma(2)} \otimes \cdots \otimes \mathbf{u}_{\sigma(n)}) = \mathbf{L} \cdot (\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \ldots \otimes \mathbf{u}_n) \tag{2.16}$$

for any $\mathbf{u}_1, \mathbf{u}_2, \ldots$ and $\mathbf{u}_n$ belonging to $\mathscr{V}$. For example, if σ is a permutation such that $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(k) = k$ for $k \in \{3, 4, \ldots, n\}$, the corresponding index permutation symmetry means that the components of $\mathbf{L}$ have the property that $L_{i_1 i_2 i_3 \ldots i_n} = L_{i_2 i_1 i_3 \ldots i_n}$.

The number of the independent components of a 2-D nth-order tensor $\mathbf{L}$ is the dimension of $\mathscr{L}$. When $\mathbf{L}$ has not any index permutation symmetry, this number equals 2^n, so that $\dim(\mathscr{L}) = 2^n$. Concerning the corresponding $\mathbf{S}$ which has all the possible index permutation symmetries, the number of its independent components becomes that of the combinations obtained by filling a table of $n (\geqslant 2)$ boxes with the elements 1 and 2. A simple calculation shows that this number equals $n+1$, so that $\dim(\mathscr{S}) = n+1$. The dimension of $\mathscr{A}$ can be deduced from the classical algebraic formula $\dim(\mathscr{L}) = \dim(\mathscr{S}) + \dim(\mathscr{A})$ for a direct sum. In brief, we have

$$\dim(\mathscr{S}) = n+1, \quad \dim(\mathscr{A}) = 2^n - n - 1, \quad \dim(\mathscr{L}) = \dim(\mathscr{S}) + \dim(\mathscr{A}) = 2^n. \tag{2.17}$$

We proceed now to deal with the decomposition of 3rd- and 4th-order tensors with certain index permutation symmetries by using (2.13), (2.15) and (2.17). First, consider the subspace $\mathscr{h}$ of all 3rd-order tensors with the index permutation symmetry σ characterized by $\sigma(\{1, 2, 3\}) = \{1, 3, 2\}$; that is, the components h_{ijm} of $\mathbb{h} \in \mathscr{h} \subset \mathbb{l}$in have the property that

$$h_{ijm} = h_{imj}. \tag{2.18}$$

Let us set $\mathbf{L} \equiv \mathbb{h}$, $\mathbf{S} \equiv \mathbb{s}$ and $\mathbf{A} \equiv \mathbb{a}$. Then, applying (2.13) while taking (2.18) into account, we obtain the components of $\mathbb{s}$ and $\mathbb{a}$ in terms of those of $\mathbb{h}$:

$$s_{ijm} = \tfrac{1}{3}(h_{ijm} + h_{jim} + h_{mij}), \quad a_{ijm} = \tfrac{1}{3}(2h_{ijm} - h_{jim} - h_{mij}). \tag{2.19}$$

Further, writing out (2.19) while bearing in mind that $\mathbb{a}$ has the index permutation symmetry of $\mathbb{h}$, we get

$$a_{111} = a_{222} = 0, \quad a_{122} = -2a_{221} = 2(h_{122} - h_{221})/3, \quad a_{211} = -2a_{112} = 2(h_{211} - h_{112})/3,$$
$$s_{111} = h_{111}, \quad s_{222} = h_{222}, \quad s_{112} = -s_{222} = 2(h_{211} - h_{112})/3,$$
$$s_{221} = -2s_{112} = 2(h_{122} - h_{221})/3.$$

By means of the tensor products defined in (2.1a), these expressions can be written in the compact form:

$$\mathbb{a} = \mathbf{v} \otimes \mathbf{1} - \tfrac{1}{2}(\mathbf{1} \otimes \mathbf{v} + \mathbf{1} \underline{\otimes} \mathbf{v}), \quad \mathbb{s} = \mathbb{h} - \mathbf{v} \otimes \mathbf{1} + \tfrac{1}{2}(\mathbf{1} \otimes \mathbf{v} + \mathbf{1} \underline{\otimes} \mathbf{v}), \tag{2.20a}$$

with

$$\mathbf{v} = \tfrac{2}{3}(h_{122} - h_{221})\mathbf{e}_1 + \tfrac{2}{3}(h_{211} - h_{112})\mathbf{e}_2. \tag{2.20b}$$

Putting $\mathscr{L} \equiv \mathscr{h}$, $\mathscr{S} \equiv \mathscr{s}$ and $\mathscr{A} \equiv \mathscr{a}$, formula (2.15) applies so that

$$\mathscr{h} = \mathscr{s} \oplus \mathscr{a}. \tag{2.21}$$

On the other hand, due to (2.18), (2.17) cannot be directly used. However, with the help of (2.20) and the simple fact that $\dim(\mathscr{h}) = 6$, we see that $\dim(\mathscr{a}) = 2$ and $\dim(\mathscr{s}) = 4$.

Next, we come to the decomposition of a 2-D elastic tensor $\mathbb{E}$. By definition, $\mathbb{E}$ is a linear mapping from Sym into itself. Denoting by $\mathscr{E}$ the subspace of $\mathbb{L}$in formed by all elastic tensors, then any $\mathbb{E} \in \mathscr{E}$ has two index permutation symmetries σ and σ': $\sigma(\{1,2,3,4\}) = \{2,1,3,4\}$ and $\sigma'(\{1,2,3,4\}) = \{1,2,4,3\}$. In more familiar words, the components E_{ijmn} of $\mathbb{E}$ have two minor symmetries:

$$E_{ijmn} = E_{jimn} = E_{ijnm}. \tag{2.22}$$

In coordinate-free notation, (2.22) reads as $\mathbb{E} = (\mathbf{1} \overline{\otimes} \mathbf{1})\mathbb{E} = \mathbb{E}(\mathbf{1} \overline{\otimes} \mathbf{1})$. Along the line taken for $\mathscr{h}$ while keeping (2.22) in mind, we can decompose $\mathscr{E}$ into the two orthogonal subspaces, $\mathscr{S}$ and $\mathscr{A}$, of symmetric and antisymmetric elastic tensors. Precisely, with a given $\mathbb{E} \in \mathscr{E}$ is associated a unique $\mathbb{S} \in \mathscr{S}$ and a unique $\mathbb{A} \in \mathscr{A}$, whose components are given by

$$S_{ijmn} = \tfrac{1}{6}(E_{ijmn} + E_{imjn} + E_{inmj} + E_{mnij} + E_{jnim} + E_{mjin}), \tag{2.23a}$$

$$A_{ijmn} = \tfrac{1}{6}(5E_{ijmn} - E_{imjn} - E_{inmj} - E_{mnij} - E_{jnim} - E_{mjin}). \tag{2.23b}$$

These two expressions are obtained by applying (2.13) to E_{ijmn} together with (2.22). As has been done for (2.19), upon writing (2.23) for $i, j, m, n = 1, 2$ and carefully examining the component expressions obtained, we can deduce that

$$\mathbb{A} = \tfrac{1}{2}(\mathbb{E}-\mathbb{E}^T)+\alpha\mathbb{R}, \quad \mathbb{S} = \tfrac{1}{2}(\mathbb{E}+\mathbb{E}^T)-\alpha\mathbb{R}, \tag{2.24a}$$

where the transposition $\mathbb{E}^T$ of $\mathbb{E}$ is understood to be such that $\mathbb{E}^T\mathbf{U}\cdot\mathbf{V} = \mathbf{U}\cdot\mathbb{E}\mathbf{V}$ for any $\mathbf{U}$, $\mathbf{V}\in\text{Sym}$, and the scalar α is given by

$$\alpha = \tfrac{1}{3}(E_{1122}+E_{2211}-2E_{1212}). \tag{2.24b}$$

As in (2.21), we can write

$$\mathscr{E} = \mathscr{S}\oplus\mathscr{A}, \tag{2.25}$$

with $\dim(\mathscr{S}) = 5$, $\dim(\mathscr{A}) = 4$ and $\dim(\mathscr{E}) = 9$.

As will be seen in Section 3, we need further decompose $\mathbb{S}\in\mathscr{S}$ into the sum of $\hat{\mathbb{S}}\in\mathscr{S}$ and $\check{\mathbb{S}}\in\mathscr{S}$, such that

$$\hat{S}_{iimm} = 0, \quad \check{S}_{iimm} = S_{iimm}. \tag{2.26}$$

In seeking the expressions of $\hat{\mathbb{S}}$ and $\check{\mathbb{S}}$ in terms of $\mathbb{S}$, it is important to observe that $S_{iimm} = \mathbb{S}\cdot(\mathbf{1}\otimes\mathbf{1}) = \mathbb{S}\cdot(\mathbf{1}\underline{\otimes}\mathbf{1}) = \mathbb{S}\cdot(\mathbf{1}\overline{\otimes}\mathbf{1})$ owing to the index permutation symmetries of $\mathbb{S}$. Then, bearing in mind (2.26) and the requirement that $\hat{\mathbb{S}}$ and $\check{\mathbb{S}}$ be symmetric, we are led to

$$\check{\mathbb{S}} = \frac{\tau}{8}(\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\underline{\otimes}\mathbf{1}+\mathbf{1}\overline{\otimes}\mathbf{1}), \quad \hat{\mathbb{S}} = \mathbb{S}-\frac{\tau}{8}(\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\underline{\otimes}\mathbf{1}+\mathbf{1}\overline{\otimes}\mathbf{1}), \tag{2.27a}$$

with

$$\tau = S_{iimm} = E_{1111}+E_{2222}+\tfrac{1}{3}(E_{1122}+E_{2211})+\tfrac{4}{3}E_{1212}. \tag{2.27b}$$

We can easily verify that $\check{\mathbb{S}}$ and $\hat{\mathbb{S}}$ are orthogonal, i.e., $\check{\mathbb{S}}\cdot\hat{\mathbb{S}} = 0$. Denoting by $\check{\mathscr{S}}$ and $\hat{\mathscr{S}}$ the respective subspaces formed by all tensors $\check{\mathbb{S}}$ and $\hat{\mathbb{S}}$, now we can write

$$\mathscr{S} = \hat{\mathscr{S}}\oplus\check{\mathscr{S}}. \tag{2.28}$$

From (2.27) and (2.28), we deduce that $\dim(\hat{\mathscr{S}}) = 4$ and $\dim(\check{\mathscr{S}}) = 1$. Introducing (2.28) into (2.25), yields

$$\mathscr{E} = \hat{\mathscr{S}}\oplus\check{\mathscr{S}}\oplus\mathscr{A}. \tag{2.29}$$

Finally, we make the decomposition of a hyperelastic tensor by specializing (2.24) and (2.27). By definition, a hyperelastic tensor $\mathbb{H}$ refers to a self-adjoint linear mapping from Sym into itself. The space formed by all hyperelastic tensors will be designated by $\mathscr{H}$. In comparison with an elastic tensor, $\mathbb{H}$ has the additional index permutation symmetry σ'' defined by $\sigma''(\{1,2,3,4\}) = \{3,4,1,2\}$. It is easy to verify that in this case, $\sigma' = \sigma''\sigma\sigma''$. So, the defining property of $\mathbb{H}$ is that its components H_{ijmn} have the following minor and major symmetries:

$$H_{ijmn} = H_{jimn} = H_{mnij}. \tag{2.30}$$

In coordinate-free notation, (2.30) reads as $\mathbb{H} = (\mathbf{1}\overline{\otimes}\mathbf{1})\mathbb{H} = \mathbb{H}^T$. Substituting $\mathbb{H}$ for $\mathbb{E}$ and using $\mathbb{H} = \mathbb{H}^T$ in (2.24), we obtain

$$\mathbb{A} = \alpha\mathbb{R}, \quad \mathbb{S} = \mathbb{H} - \alpha\mathbb{R}, \quad \alpha = \tfrac{2}{3}(H_{1122} - H_{1212}). \tag{2.31}$$

Upon setting $\mathscr{E} \equiv \mathscr{H}$, (2.25) remains valid but $\dim(\mathscr{S}) = 5$, $\dim(\mathscr{A}) = 1$ and $\dim(\mathscr{H}) = 6$. Furthermore, (2.27)–(2.29) hold provided that $\mathscr{E}$ is replaced by $\mathscr{H}$, and (2.27b) by

$$\tau = H_{1111} + H_{2222} + \tfrac{2}{3}H_{1122} + \tfrac{4}{3}H_{1212}. \tag{2.32}$$

The formulae in (2.31) have already been given in (He, 1997), but here they have been deduced from the more general formulae (2.24).

3. NECESSARY AND SUFFICIENT CONDITIONS FOR THE INVARIANCE OF PLANE STRESSES

With the notation and preliminary results presented in the previous section, we are now ready to formulate the problem described in the Introduction and to prove step by step the two main assertions stated there.

3.1. *Formulation of the stress invariance problem*

In order to focus attention on the basic ideas underlying our investigations, we shall consider only those bodies which occupy bounded simply-connected domains with piecewise smooth boundaries; in addition, any field over the domain $\bar{\Omega}$ occupied by such a solid will be assumed to be sufficiently regular so that its derivatives involved make sense. Removal of the restriction concerning the simple connection of $\bar{\Omega}$ can be made along the line taken by Dundurs and Markenscoff (1993), and relaxation of the hypothesis relative to the continuity of elastic tensor fields can be carried out on the basis of the important remark made by Milton and Movchan (1995) at the end of Section 5 of their paper.

As in Cherkaev *et al.* (1992), the traction boundary problem of the plane linear problem is below formulated in terms of the classical Airy function. First, consider the case (a) of Fig. 1, where the 2-D (infinitesimal) strain and stress tensor fields, $\mathbf{E}(\mathbf{x})$ and $\mathbf{T}(\mathbf{x})$, are related according to Hooke's law:

$$\mathbf{E}(\mathbf{x}) = \mathbb{K}(\mathbf{x})\mathbf{T}(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}. \tag{3.1}$$

Here, the 4th-order tensor $\mathbb{K}$ has at least two minor symmetries resulting from the symmetry of $\mathbf{E}$ and that of $\mathbf{T}$. In other words, $\mathbb{K}$ belongs to the elastic tensor space $\mathscr{E}$. When (3.1) is derived from a scalar function, $\mathbb{K}$ possesses the additional major symmetry for it to belong to the hyperelastic tensor space $\mathscr{H}$. In our forthcoming formulation, the condition that $\mathbb{K} \in \mathscr{H}$ is not required. Let $\varphi : \bar{\Omega} \to R$ be the Airy function, which is assumed to be of class $C^{(4)}$ on the interior Ω of $\bar{\Omega}$ and piecewise twice continuously differentiable on the boundary $\partial\Omega$ of $\bar{\Omega}$. Then the stress field derived from φ, i.e.,

$$\mathbf{T}(\mathbf{x}) = \mathbb{R}(\nabla \otimes \nabla)\varphi(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega} \tag{3.2}$$

where $\mathbb{R}$ is given by (1.1_1) and $\nabla \otimes \nabla$ is defined by (2.10a), always fulfills the equation of equilibrium in the absence of body forces:

$$\mathbf{T}(\mathbf{x})\nabla = T_{ij,j}(\mathbf{x})\mathbf{e}_i = \mathbf{0}, \quad \mathbf{x} \in \Omega. \tag{3.3}$$

With the hypothesis that $\bar{\Omega}$ is simply connected, the strain field $\mathbf{E}(\mathbf{x})$ obtained from (3.1) can be derived from a displacement field $\mathbf{u} : \Omega \to \mathscr{V}$, i.e.,

$$\mathbf{E}(\mathbf{x}) = \tfrac{1}{2}[\nabla\mathbf{u}(\mathbf{x}) + (\nabla\mathbf{u}(\mathbf{x}))^T], \quad \mathbf{x} \in \Omega, \tag{3.4}$$

if and only if $\mathbf{E}(\mathbf{x})$ satisfies the compatibility equation:

$$(\nabla \otimes \nabla) \cdot [\mathbb{R}\mathbf{E}(\mathbf{x})] = 0, \quad \mathbf{x} \in \Omega. \tag{3.5}$$

In addition, $\mathbf{T}(\mathbf{x})$ must satisfy the prescribed traction boundary condition:

$$\mathbf{T}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3.6}$$

where $\mathbf{n}(\mathbf{x})$ designates the unit outward normal to $\partial\Omega$ at $\mathbf{x}$ and the surface traction vector function $\mathbf{t}: \partial\Omega \to \mathscr{V}$ is assumed to be piecewise continuous and self-equilibrated. Combining (3.2), (3.1) and (3.5) and introducing (3.2) into (3.6), we obtain the classical formulation of the traction boundary value problem in terms of φ:

$$(P)\begin{cases} (\nabla \otimes \nabla) \cdot [\mathbb{R}\mathbb{K}(\mathbf{x})\mathbb{R}(\nabla \otimes \nabla)\varphi(\mathbf{x})] = 0, \quad \mathbf{x} \in \Omega, & (3.7) \\ [\mathbb{R}(\nabla \otimes \nabla)\varphi(\mathbf{x})]\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. & (3.8) \end{cases}$$

Next, consider the case (a′) of Fig. 1, which is different from the case (a) only in that the strain and stress fields, $\mathbf{E}'(\mathbf{x})$ and $\mathbf{T}'(\mathbf{x})$, are related by another elastic tensor field $\mathbb{K}'(\mathbf{x})$:

$$\mathbf{E}'(\mathbf{x}) = \mathbb{K}'(\mathbf{x})\mathbf{T}'(\mathbf{x}) = [\mathbb{K}(\mathbf{x}) + \mathbb{D}(\mathbf{x})]\mathbf{T}'(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \tag{3.9}$$

where (1.3) is employed. Let $\varphi': \bar{\Omega} \to R$ be the corresponding Airy function, then the stress field is given by

$$\mathbf{T}'(\mathbf{x}) = \mathbb{R}(\nabla \otimes \nabla)\varphi'(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}. \tag{3.10}$$

Analogous to the case (a), the formulation of the traction boundary value problem in the case (a′) takes the form

$$(P')\begin{cases} (\nabla \otimes \nabla) \cdot [\mathbb{R}\mathbb{K}(\mathbf{x})\mathbb{R}(\nabla \otimes \nabla)\varphi'(\mathbf{x})] \\ \qquad + (\nabla \otimes \nabla) \cdot [\mathbb{R}\mathbb{D}(\mathbf{x})\mathbb{R}(\nabla \otimes \nabla)\varphi'(\mathbf{x})] = 0, \quad \mathbf{x} \in \Omega, & (3.11) \\ [\mathbb{R}(\nabla \otimes \nabla)\varphi'(\mathbf{x})]\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. & (3.12) \end{cases}$$

Hereafter, we suppose that the boundary value problem data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ (or $(\bar{\Omega}, \mathbb{K}', \mathbf{t})$) are such that (P) (or (P')) admits a sufficiently regular solution ψ (or ψ'), which is unique modulo an affine function of $\mathbf{x}$. It is in this sense that we shall speak of the unique solution of (P) or (P'). Moreover, a distinction will be made between the unknown and solution of each problem by using two different symbols, say, φ for the unknown of (P) and ψ for its solution.

In view of (3.2) and (3.10), the requirement that $\mathbf{T}'(\mathbf{x})$ be the same as $\mathbf{T}(\mathbf{x})$ for any $\mathbf{x} \in \bar{\Omega}$ implies that the difference between ψ and ψ' is at most an affine function of $\mathbf{x}$:

$$\psi'(\mathbf{x}) - \psi(\mathbf{x}) = p + \mathbf{q} \cdot \mathbf{x}, \quad \mathbf{x} \in \bar{\Omega}, \tag{3.13}$$

where p is a constant scalar and $\mathbf{q} \in \mathscr{V}$ is a constant vector. Introducing (3.13) into (3.11) and using the fact that ψ satisfies equation (3.7), we obtain

$$(\nabla \otimes \nabla) \cdot [\mathbb{R}\mathbb{D}(\mathbf{x})\mathbb{R}(\nabla \otimes \nabla)\psi(\mathbf{x})] = 0, \quad \mathbf{x} \in \Omega. \tag{3.14}$$

This is the necessary and sufficient condition that $\mathbb{D}(\mathbf{x})$ must satisfy in order that $\mathbf{T}'(\mathbf{x})$ coincides with $\mathbf{T}(\mathbf{x})$ at each $\mathbf{x} \in \bar{\Omega}$. In passing, note that, since $\mathbb{D}$ represents the difference between $\mathbb{K}'$ and $\mathbb{K}$ belonging both to $\mathscr{E}$, it has generally not the major symmetry.

It is important to remember that in (3.14), $\psi(\mathbf{x})$ is the solution of problem (P) and depends on the data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ for that problem. Consequently, of the set formed by all $\mathbb{D}$'s satisfying (3.14), there may be some elements which are dependent on $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$. As explained in the Introduction, we are, however, interested only in those elastic tensor fields $\mathbb{D}(\mathbf{x})$ which verify (3.14) regardless of $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$. More precisely, our problem here is to find the 4th-order tensor valued functions $\mathbb{D}(\mathbf{x})$ with the minor symmetries, such that (3.14) holds for each element ψ of the set Ψ of solutions of problem (P) obtained by varying data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ in all possible ways. Thus, the direct approach to our problem is to introduce each element ψ of Ψ into (3.14) and to solve the resulting partial differential equations for $\mathbb{D}$. However, this approach can be used only when the set Ψ is entirely available. Unfortunately, Ψ is currently unknown, and is very difficult to characterize since data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ can be varied in an infinite number of manners. The strategy we elaborate for overcoming this difficulty consists of:

(i) determining certain simple elements of Ψ or equivalently the solutions of some traction boundary value problems (P) with simple data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$;
(ii) introducing each of the previously determined particular elements of Ψ into (3.14) and solving the resulting equation about $\mathbb{D}(\mathbf{x})$ so as to establish a certain number of necessary conditions for $\mathbb{D}(\mathbf{x})$;
(iii) verifying whether the necessary conditions lastly constructed for $\mathbb{D}(\mathbf{x})$ are also sufficient, i.e., whether the form of $\mathbb{D}(\mathbf{x})$ resulting from (ii) is such that equations (3.14) is satisfied regardless of ψ.

If the answer from (iii) is positive, then the form of $\mathbb{D}(\mathbf{x})$ obtained is both necessary and sufficient for preserving the invariance of stresses. It is worth making the two remarks on this strategy. Firstly, the problems (P) with particular data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ must be well-posed, i.e., that each of them has one and only one solution $\psi(\mathbf{x})$ to within an affine function of $\mathbf{x}$, because the function ψ in eqn (3.14) is the solution of such a problem. Secondly, the fact that only some particular traction boundary value problems are used in step (ii) causes no loss of generality if the necessary conditions deduced from (ii) are verified to be sufficient in step (iii).

When treating (3.14), it is convenient to define

$$\mathbb{C}(\mathbf{x}) := \mathbb{R}\mathbb{D}(\mathbf{x})\mathbb{R}, \tag{3.15}$$

and to write (3.14) in the following equivalent form:

$$(\nabla \otimes \nabla) \cdot [\mathbb{C}(\mathbf{x})(\nabla \otimes \nabla)\psi(\mathbf{x})] = 0, \quad \mathbf{x} \in \Omega. \tag{3.16}$$

It is immediate from (3.15) that $\mathbb{C}$ possesses the minor symmetries. Once $\mathbb{C}$ is known, we can use the property (2.7) of $\mathbb{R}$ for inverting (3.15) and obtain

$$\mathbb{D}(\mathbf{x}) = \mathbb{R}\mathbb{C}(\mathbf{x})\mathbb{R}. \tag{3.17}$$

Therefore, (3.14) will be replaced by (3.16) together with (3.17).

Before using the strategy described above to find the general form of $\mathbb{D}(\mathbf{x})$, it seems to us important to emphasize the hypothesis that either the problem (P) or (P') has a unique smooth solution to within an affine function of $\mathbf{x}$. This hypothesis, tacitly made by Cherkaev *et al.* (1992) and other authors, is essential, because the comparison between the fields $\mathbf{T}(\mathbf{x})$ and $\mathbf{T}'(\mathbf{x})$ makes sense only when each of them exists and is unique. The restrictions imposed by this requirement on the data $(\bar{\Omega}, \mathbb{K}, \mathbf{t})$ and $(\bar{\Omega}, \mathbb{K}', \mathbf{t})$ are beyond the scope of the present work. For this, the reader can refer to Knops and Payne (1971) and Duvaut (1990). Nevertheless, it is useful to note that once the problem (P) has a unique solution, so does the problem (P') provided (3.14) is ensured.

3.2. *Solution of the stress invariance problem in the hyperelastic case*

In this paragraph, $\mathbb{D}(\mathbf{x})$ is assumed to be hyperelastic. By (3.15), this amounts to assuming that $\mathbb{C} \in \mathscr{H}$. So, $\mathbb{C}(\mathbf{x})$ has the following index permutation symmetries:

$$C_{ijmn}(\mathbf{x}) = C_{jimn}(\mathbf{x}) = C_{mnij}(\mathbf{x}). \tag{3.18}$$

Then, the results on hyperelastic tensors, derived in Section 2.2, apply. As a consequence of (2.31), we can write

$$\mathbb{C}(\mathbf{x}) = \mathbb{S}(\mathbf{x}) + \alpha(\mathbf{x})\mathbb{R}, \quad \alpha(\mathbf{x}) = \tfrac{2}{3}[C_{1122}(\mathbf{x}) - C_{1212}(\mathbf{x})]. \tag{3.19}$$

Introducing (3.19_1) into (3.16) gives

$$(\nabla \otimes \nabla)\cdot[\mathbb{S}(\mathbf{x})(\nabla \otimes \nabla)\psi(\mathbf{x})] + (\nabla \otimes \nabla)\cdot[\alpha(\mathbf{x})\mathbb{R}(\nabla \otimes \nabla)\psi(\mathbf{x})] = 0. \tag{3.20}$$

Use of the orthogonality relations (He, 1997)

$$\mathbb{R}\cdot(\nabla \otimes \nabla \otimes \nabla \otimes \nabla)\psi = 0, \quad \mathbb{R}\cdot[\nabla\alpha \otimes (\nabla \otimes \nabla \otimes \nabla)\psi] = 0, \tag{3.21}$$

in developing the 2nd term of the right-hand member of (3.20) allows us to write it as

$$(\nabla \otimes \nabla)\cdot[\mathbb{S}(\mathbf{x})(\nabla \otimes \nabla)\psi(\mathbf{x})] + \Delta\alpha(\mathbf{x})\Delta\psi(\mathbf{x}) - [(\nabla \otimes \nabla)\alpha(\mathbf{x})]\cdot[(\nabla \otimes \nabla)\psi(\mathbf{x})] = 0, \quad \mathbf{x}\in\Omega, \tag{3.22}$$

where Δ is the usual Laplacian operator.

Hence, the initial problem of determining $\mathbb{C}(\mathbf{x})$ such that (3.16) is fulfilled for any $\psi \in \Psi$ has been transformed into that of finding $\alpha(\mathbf{x})$ and $\mathbb{S}(\mathbf{x})$ so that (3.22) is satisfied for all $\psi \in \Psi$. The solution of this problem is contained in the following statement.

Proposition 3.1. Equation (3.22) holds for the solution ψ of any traction boundary value problem (P) if and only if

$$\alpha(\mathbf{x}) = \mathbf{b}\cdot\mathbf{x} + c, \quad \mathbb{S}(\mathbf{x}) = \mathbb{O}. \tag{3.23}$$

Proof. Sufficiency is immediate. It remains to show necessity. To this end, we write out (3.22) in index notations while omitting the variable $\mathbf{x}$ for simplicity:

$$S_{ijmn,ij}\psi_{,mn} + 2S_{ijmn,i}\psi_{,jmn} + S_{ijmn}\psi_{,ijmn} + \alpha_{,ii}\psi_{,mm} - \alpha_{,mn}\psi_{,mn} = 0. \tag{a}$$

Recall that, according to the Schwarz theorem, $\psi_{,mn}$, $\psi_{,jmn}$ and $\psi_{,ijmn}$ are unchanged with any index permutation.

The first traction boundary value problem used to establish a necessary condition for $\alpha(\mathbf{x})$ and $\mathbb{S}(\mathbf{x})$ is

$$(P_1)\begin{cases}\Delta\Delta\varphi(\mathbf{x}) = 0 & \text{if } \|\mathbf{x}\| < 1,\\ [\mathbb{R}(\nabla \otimes \nabla)\varphi(\mathbf{x})]\mathbf{x} = (\mathbb{R}\mathbf{F})\mathbf{x} & \text{if } \|\mathbf{x}\| = 1,\end{cases}$$

where $\Delta\Delta$ denotes the biharmonic operator and $\mathbf{F}$ is an arbitrary non-zero constant 2nd-order symmetric tensor. Problem (P_1) corresponds to problem (P) with $\bar{\Omega}$ being a disc of unit radius, $\mathbb{K}(\mathbf{x})$ a constant isotropic hyperelastic tensor and $\mathbf{t}(\mathbf{x}) = \mathbb{R}\mathbf{F}\mathbf{x}$. Let us first check that the surface tractions $\mathbf{t}$ on $\partial\Omega = \{\mathbf{x} \mid \|\mathbf{x}\| = 1\}$ are self-equilibrated. Indeed, by putting $\mathbf{T} = \mathbb{R}\mathbf{F}$ and noting the $\mathbf{n}(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \partial\Omega$, we have the force and moment equilibrium:

$$\int_{\partial\Omega} (\mathbf{t})_i \,\mathrm{d}s = \int_{\partial\Omega} T_{ij}x_j \,\mathrm{d}s = \int_{\partial\Omega} T_{ij}n_j \,\mathrm{d}s = \int_{\Omega} T_{ij,j} \,\mathrm{d}a = 0,$$

$$\int_{\partial\Omega} (\mathbf{x}\times\mathbf{t})_i \,\mathrm{d}s = \int_{\partial\Omega} \varepsilon_{ijk}x_jt_k \,\mathrm{d}s = \int_{\partial\Omega} \varepsilon_{ijk}x_jT_{km}n_m \,\mathrm{d}s = \int_{\Omega} \varepsilon_{ijk}T_{kj} \,\mathrm{d}a = 0,$$

where ε_{ijk} are the components of the usual permutation tensor, and the divergence theorem has been employed. Next, it is easy to verify that

$$\psi(\mathbf{x}) = \tfrac{1}{2}\mathbf{F}\cdot(\mathbf{x}\otimes\mathbf{x}) \tag{b}$$

is one solution of (P_1). On the other hand, it is known (see e.g. Muskhelishivili, 1963; Knops and Payne, 1971) that such a problem admits at most one solution to within an affine function of $\mathbf{x}$. Therefore, the function $\psi(\mathbf{x})$ defined by (b) is effectively an element of Ψ. Inserting (b) into (a) while noting that all higher derivatives of $\psi(\mathbf{x})$ than the 2nd one are zero, it follows from the arbitrariness of $\mathbf{F}$ that

$$S_{ijmn,ij}+\alpha_{,ii}\delta_{mm}-\alpha_{,mn} = 0, \tag{c}$$

which, in its turn, reduces (a) to

$$2S_{ijmn,i}\psi_{,jmn}+S_{ijmn}\psi_{,ijmn} = 0. \tag{d}$$

Next, we modify problem (P_1) upon replacing the homogeneous traction boundary condition by a non homogenous one:

$$(P_2)\begin{cases}\Delta\Delta\varphi(\mathbf{x}) = 0 & \text{if } \|\mathbf{x}\| < 1,\\ [\mathbb{R}(\nabla\otimes\nabla)\varphi(\mathbf{x})]\mathbf{x} = [\mathbb{R}(\mathbb{f}\mathbf{x})]\mathbf{x} & \text{if } \|\mathbf{x}\| = 1.\end{cases}$$

Here, $\mathbb{f}\in\mathscr{J}$ is an arbitrary non-zero constant 3rd-order symmetric tensor and $(\mathbb{f}\mathbf{x})_{ij} = f_{ijm}x_m$. As in (P_1), to within an affine function of $\mathbf{x}$ the problem (P_2) has the unique solution

$$\psi(\mathbf{x}) = \tfrac{1}{6}\mathbb{f}\cdot(\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}). \tag{e}$$

Introducing (e) into (d), the arbitrariness of $\mathbb{f}$ allows us to conclude that

$$S_{ijmn,i} = 0. \tag{f}$$

In view of this, (d) can be simplified into

$$S_{ijmn}\psi_{,ijmn} = 0. \tag{g}$$

In addition, (f) implies that $S_{ijmn,ij} = 0$, so that (c) becomes

$$\alpha_{,ii}\delta_{mn}-\alpha_{,mn} = 0. \tag{h}$$

Writing out (h) component by component, we see

$$\alpha_{,ij} = 0 \quad (i,j = 1,2). \tag{i}$$

This means that $\alpha(\mathbf{x})$ is an affine function of $\mathbf{x}$, thus proving the necessity of $(3.23)_1$.

To further use the necessary condition (g), consider

$$(P_3)\begin{cases}\Delta\Delta\varphi(\mathbf{x}) = 0 & \text{if } \|\mathbf{x}\| < 1,\\ [\mathbb{R}(\nabla\otimes\nabla)\varphi(\mathbf{x})]\mathbf{x} = [\mathbb{R}\hat{\mathbb{F}}(\mathbf{x}\otimes\mathbf{x})]\mathbf{x} & \text{if } \|\mathbf{x}\| = 1,\end{cases}$$

where $\hat{\mathbb{F}}\in\hat{\mathscr{S}}$ is an arbitrary non-zero constant 4th-order symmetric tensor and has the additional property that

$$\hat{F}_{iimm} = \tfrac{1}{3}(\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \underline{\otimes} \mathbf{1} + \mathbf{1} \overline{\otimes} \mathbf{1}) \cdot \hat{\mathbb{F}} = 0. \tag{j}$$

Similar to the preceding problems, (P_3) has a unique solution modulo an affine function of $\mathbf{x}$:

$$\psi(\mathbf{x}) = \tfrac{1}{12}\hat{\mathbb{F}} \cdot (\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}). \tag{k}$$

Indeed, the property (j) and the identity

$$\Delta\Delta\varphi = \tfrac{1}{3}(\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \underline{\otimes} \mathbf{1} + \mathbf{1} \overline{\otimes} \mathbf{1}) \cdot (\nabla \otimes \nabla \otimes \nabla \otimes \nabla)\varphi$$

ensure that $\Delta\Delta\psi = 0$. Substituting of (k) into (g) gives

$$\mathbb{S} \cdot \hat{\mathbb{F}} = 0, \tag{l}$$

which indicates that $\mathbb{S}$ is perpendicular to $\hat{\mathbb{F}}$. Since $\hat{\mathbb{F}} \in \hat{\mathscr{S}}$, it follows from (2.28) and (2.27) that $\mathbb{S} \in \check{\mathscr{S}}$ and

$$\mathbb{S} = \frac{\tau}{8}(\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \underline{\otimes} \mathbf{1} + \mathbf{1} \overline{\otimes} \mathbf{1}). \tag{m}$$

Thus, (g) becomes

$$\tau\Delta\Delta\psi(\mathbf{x}) = 0. \tag{n}$$

Moreover, (m) together with (f) implies that τ is constant.

At present, we do not know whether τ is zero or non-zero. To clarify this point, consider another traction boundary value problem:

$$(P_4)\begin{cases}\Delta\Delta\varphi(\mathbf{x}) + 2(\gamma - 1)\varphi_{,1122}(\mathbf{x}) = 0 & \text{if } \|\mathbf{x}\| < 1, \\ 2[\mathbb{R}(\nabla \otimes \nabla)\varphi(\mathbf{x})]\mathbf{x} = -(\gamma x_1^3 + x_1 x_2^2)\mathbf{e}_1 - (\gamma x_2^3 + x_1^2 x_2)\mathbf{e}_2 & \text{if } \|\mathbf{x}\| = 1,\end{cases}$$

where γ is a scalar constant. This problem is the problem (P) in which $\bar{\Omega}$ is a disc of unit radius, $\mathbb{K}$ is homogenous square-symmetric with $K_{1111} > 0$ and $(K_{1122} + 2K_{1212})/K_{1111} = \gamma$ [see He (1997)] and the surface tractions are given by $\mathbf{t}(\mathbf{x}) = -(\gamma x_1^3 + x_1 x_2^2)\mathbf{e}_1 - (\gamma x_2^3 + x_1^2 x_2)\mathbf{e}_2$. For $\mathbb{K}$ not to degenerate into an isotropic elastic tensor, we require that $\gamma \neq 1$. To within an affine function of $\mathbf{x}$, the unique solution of (P_4) is

$$\psi(\mathbf{x}) = -\tfrac{1}{24}\gamma(x_1^4 + x_2^4) + \tfrac{1}{4}x_1^2 x_2^2.$$

As $\psi(\mathbf{x})$ satisfies the first equation of (P_4),

$$\Delta\Delta\psi(\mathbf{x}) = 2(1-\gamma)\psi_{,1122}(\mathbf{x}) = 2(1-\gamma). \tag{o}$$

Recalling that $\gamma \neq 1$, we deduce from (n) and (o) that

$$\tau = 0. \tag{p}$$

Substituting this result into (m) proves the necessity of $(3.23)_2$.

In the preceding proof, the solutions of problems (P_1)–(P_4) have played a crucial role. This may give us the impression that the conclusion obtained would depend on these particular solutions and, hence, be not general. In reality, the necessary conditions for $\alpha(\mathbf{x})$ and $\mathbb{S}(\mathbf{x})$ derived by using these solutions have turned out to be also sufficient, i.e. (3.22) holds irrespective of ψ. Thus, the generality of the conclusion is not affected by use of the particular problems (P_1)–(P_4). Another point which should be emphasized is the verification of uniqueness of each solution. Indeed, the function $\psi(\mathbf{x})$ in eqn (3.14) [or (3.16)] is the solution of a traction boundary value problem of plane linear elasticity, which is assumed to unique to within an affine function of $\mathbf{x}$. For uniqueness of the solutions of problems (P_1)–(P_4) we have referred to the criteria given in Knops and Payne (1971).

Proposition 3.1 together with $(3.19)_1$ tells us that the most general form for $\mathbb{C}(\mathbf{x})$ is

$$\mathbb{C}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R}. \tag{3.24}$$

Applying (3.17) and using the property (2.7) of $\mathbb{R}$, we get the form of $\mathbb{D}(\mathbf{x})$:

$$\mathbb{D}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R} \quad \text{or } D_{ijmn}(\mathbf{x}) = (b_k x_k + c)R_{ijmn}. \tag{3.25}$$

Thus, we have completed the proof of the following result.

Theorem 3.2. Let $\mathscr{B}$ be a simply-connected plane solid which is made of a linearly elastic material M characterized by an elastic tensor field $\mathbb{K}(\mathbf{x})$, and is subjected to prescribed traction forces on its boundary. Consider another linearly elastic material M' which is characterized by an elastic tensor field $\mathbb{K}'(\mathbf{x})$ different from $\mathbb{K}(\mathbf{x})$ by some hyperelastic tensor field, and is used to substitute for M. Then the stress tensor field of $\mathscr{B}$ is unchanged with the substitution of M' for M if and only if $\mathbb{K}'(\mathbf{x})$ differs from $\mathbb{K}(\mathbf{x})$ by $\mathbb{D}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R}$.

Compared with the results of Cherkaev *et al.* (1992) and Dundurs and Markenscoff (1993), this theorem is stronger in the sense that it affirms the necessity of relation (1.2). Moreover, note that neither $\mathbb{K}(\mathbf{x})$ nor $\mathbb{K}'(\mathbf{x})$ is assumed to have the major symmetry, but only their difference is required to have such a property.

3.3. *Solution of the stress invariance problem in the elastic case*

We turn to (3.16) and seek the general form of $\mathbb{C}(\mathbf{x})$ when it has only the minor symmetries:

$$C_{ijmn}(\mathbf{x}) = C_{jimn}(\mathbf{x}) = C_{ijnm}(\mathbf{x}). \tag{3.26}$$

The results of paragraph 2.2 [eqns (2.24)–(2.25)] concerning the decomposition of an elastic tensor allows us to partition $\mathbb{C}(\mathbf{x})$ in the following way:

$$\mathbb{C}(\mathbf{x}) = \mathbb{S}(\mathbf{x})+\mathbb{W}(\mathbf{x})+\alpha(\mathbf{x})\mathbb{R}, \tag{3.27a}$$

$$\mathbb{S}(\mathbf{x}) = \tfrac{1}{2}[\mathbb{C}(\mathbf{x})+\mathbb{C}^T(\mathbf{x})]-\alpha(\mathbf{x})\mathbb{R}, \quad \mathbb{W}(\mathbf{x}) = \tfrac{1}{2}[\mathbb{C}(\mathbf{x})-\mathbb{C}^T(\mathbf{x})], \tag{3.27b}$$

$$\alpha(\mathbf{x}) = \tfrac{1}{3}[C_{1122}(\mathbf{x})+C_{2211}(\mathbf{x})-2C_{1212}(\mathbf{x})]. \tag{3.27c}$$

Inserting (3.27a) into (3.16) yields

$$(\nabla\otimes\nabla)\cdot\{[\mathbb{S}(\mathbf{x})+\mathbb{W}(\mathbf{x})](\nabla\otimes\nabla)\psi(\mathbf{x})\}+(\nabla\otimes\nabla)\cdot[\alpha(\mathbf{x})\mathbb{R}(\nabla\otimes\nabla)\psi(\mathbf{x})] = 0. \tag{3.28}$$

Using (3.21), we can write (3.28) in the equivalent form

$$0 = (\nabla\otimes\nabla)\cdot\{[\mathbb{S}(\mathbf{x})+\mathbb{W}(\mathbf{x})](\nabla\otimes\nabla)\psi(\mathbf{x})\}+\Delta\alpha(\mathbf{x})\Delta\psi(\mathbf{x}) \\ -[(\nabla\otimes\nabla)\alpha(\mathbf{x})]\cdot[(\nabla\otimes\nabla)\psi(\mathbf{x})], \quad \mathbf{x}\in\Omega. \tag{3.29}$$

This equation is different from (3.22) only in that the antisymmetric tensor $\mathbb{W}(\mathbf{x})$ appears.

To find the general forms of $\alpha(\mathbf{x})$, $\mathbb{S}(\mathbf{x})$ and $\mathbb{W}(\mathbf{x})$ such that (3.29) is fulfilled for any $\psi\in\Psi$, we employ the same approach as that used in the hyperelastic case. However, the presence of $\mathbb{W}(\mathbf{x})$ renders it more complicated.

Proposition 3.3. Equation (3.29) holds for the solution ψ of any traction boundary value problem (P) if and only if

$$\alpha(\mathbf{x}) = \mathbf{b}\cdot\mathbf{x}+c, \quad \mathbb{S}(\mathbf{x}) = \mathbb{O}, \tag{3.30a}$$

$$W_{1111}(\mathbf{x}) = W_{2222}(\mathbf{x}) = W_{1212}(\mathbf{x}) = 0, \tag{3.30b}$$

$$W_{1122}(\mathbf{x}) = -W_{2211}(\mathbf{x}) = \omega_1+\zeta x_1+\eta x_2+2\theta x_1 x_2, \tag{3.30c}$$

$$W_{1112}(\mathbf{x}) = -W_{1211}(\mathbf{x}) = \omega_2+\eta x_1+\theta x_1^2, \tag{3.30d}$$

$$W_{2212}(\mathbf{x}) = -W_{1222}(\mathbf{x}) = \omega_3-\zeta x_2-\theta x_2^2, \tag{3.30e}$$

where ω_1, ω_2, ω_3, ζ, η and θ are constant.

Proof. It is convenient to write out (3.29) in index notation:

$$(S_{ijmn,ij}+W_{ijmn,ij})\psi_{,mn}+2(S_{ijmn,i}+W_{ijmn,i})\psi_{,jmn}+S_{ijmn}\psi_{,ijmn}+\alpha_{,ii}\psi_{,mm}-\alpha_{,mn}\psi_{,mn} = 0. \tag{a'}$$

As in proving Proposition 3.1, we now successively substitute each of the solutions of (P_1)–(P_4) into (a′) to establish the necessary condition for α, $\mathbb{S}$ and $\mathbb{W}$.

If the solution of (P_1), i.e., the expression (b), is used in (a′), we infer from the arbitrariness of $\mathbf{F}$ that

$$S_{ijmn,ij}+W_{ijmn,ij}+\alpha_{,ii}\delta_{mn}-\alpha_{,mn} = 0. \tag{c'}$$

This reduces (a′) to

$$2(S_{ijmn,i}+W_{ijmn,i})\psi_{,jmn}+S_{ijmn}\psi_{,ijmn} = 0. \tag{d'}$$

Setting $h_{jmn} = S_{ijmn,i}+W_{ijmn,i}$, then it is clear that $h_{jmn} = h_{jnm}$ and the corresponding tensor $\mathbb{h}$ belongs to the subspace $\mathscr{h}$ defined in Subsection 2.2. Then, formulae (2.20) and (2.21) apply to $\mathbb{h}$:

$$\mathbb{a} = \mathbf{v}\otimes\mathbf{1}-\tfrac{1}{2}(\mathbf{1}\otimes\mathbf{v}+\mathbf{1}\,\underline{\otimes}\,\mathbf{v}), \quad \mathbb{s} = \mathbb{h}-\mathbf{v}\otimes\mathbf{1}+\tfrac{1}{2}(\mathbf{1}\otimes\mathbf{v}+\mathbf{1}\,\underline{\otimes}\,\mathbf{v}).$$

If the solution (e) is introduced into (d′), we have

$$(S_{ijmn,i}+W_{ijmn,i})f_{jmn} = (s_{jmn}+a_{jmn})f_{jmn} = s_{jmn}f_{jmn} = 0,$$

where the 2nd equality is owing to the fact that $\mathbb{f}$ is orthogonal to $\mathbb{a}$. Since f_{jmn} is arbitrary, it follows that $s_{jmn} = 0$. This amounts to writing

$$S_{ijmn,i}+W_{ijmn,i} = a_{jmn} = v_j\delta_{mn}-\tfrac{1}{2}(\delta_{jm}v_n+\delta_{jn}v_m).$$

As $S_{ijmn,i}$ is symmetric relative to the indices i, j, m, and a_{jmn} is antisymmetric, we infer from the foregoing expression that

$$S_{ijmn,i} = 0, \tag{f'}$$

$$W_{ijmn,i} = v_j\delta_{mn}-\tfrac{1}{2}(\delta_{jm}v_n+\delta_{jn}v_m). \tag{f''}$$

These two expressions simplify (d′) into

$$S_{ijmn}\psi_{,ijmn} = 0. \tag{g'}$$

Moreover, (f) and (f″) imply that

$$S_{ijmn,ij} = 0, \quad W_{ijmn,ij} = v_{j,j}\delta_{mn} - \tfrac{1}{2}(v_{m,n}+v_{n,m}).$$

With this, (c′) becomes

$$\alpha_{,mn} - \alpha_{,ii}\delta_{mn} = W_{ijmn,ij}. \tag{h'}$$

Using condition (g′) in the same way as in the proof of Proposition 3.1, we conclude that $\mathbb{S}(\mathbf{x})$ necessarily corresponds to the 4th-order zero tensor.

The equations governing $\alpha(\mathbf{x})$ and $\mathbb{W}(\mathbf{x})$ are (h′) and (f′). To deal with these two equations, it is helpful to write out the components of $\mathbb{W}(\mathbf{x})$. By definition, $\mathbb{W}(\mathbf{x}) = [\mathbb{C}(\mathbf{x}) - \mathbb{C}^T(\mathbf{x})]/2$. So,

$$W_{1111}(\mathbf{x}) = W_{2222}(\mathbf{x}) = W_{1212}(\mathbf{x}) = 0,$$

which corresponds to (3.39b). For notational convenience the non-zero components of $\mathbb{W}(\mathbf{x})$ are denoted as follows:

$$X(\mathbf{x}) = W_{1122}(\mathbf{x}) = -W_{2211}(\mathbf{x}), \quad Y(\mathbf{x}) = W_{1112}(\mathbf{x}) = -W_{1211}(\mathbf{x}),$$
$$Z(\mathbf{x}) = W_{2212}(\mathbf{x}) = -W_{1222}(\mathbf{x}).$$

Writing out (f″), we see that the first partial derivatives of X, Y and Z are interrelated in a simple way:

$$X_{,1}(\mathbf{x}) = -Z_{,2}(\mathbf{x}), \quad X_{,2}(\mathbf{x}) = Y_{,1}(\mathbf{x}),$$
$$Y_{,2}(\mathbf{x}) = 0, \quad Z_{,1}(\mathbf{x}) = 0.$$

The last two equations imply that Y is independent of x_2 and Z of x_1. Consequently, we can write

$$Y(\mathbf{x}) = y(x_1), \quad Z(\mathbf{x}) = z(x_2).$$

In view of this, the first two equations become

$$X_{,1}(\mathbf{x}) = -\frac{\mathrm{d}z(x_2)}{\mathrm{d}x_2}, \quad X_{,2}(\mathbf{x}) = \frac{\mathrm{d}y(x_1)}{\mathrm{d}x_1}.$$

Integrating $X_{,1}(\mathbf{x})$ with respect to x_1, we obtain

$$X(\mathbf{x}) = -\frac{\mathrm{d}z(x_2)}{\mathrm{d}x_2}x_1 + g(x_2).$$

This is compatible with the fact that $X_{,2}(\mathbf{x})$ is a function of x_1 alone, if and only if

$$\frac{\mathrm{d}^2 z(x_2)}{\mathrm{d}x_2^2} = -2\theta, \quad \frac{\mathrm{d}g(x_2)}{\mathrm{d}x_2} = \eta,$$

where θ and η are two constants. Integrating these two equations gives

$$z(x_2) = -\theta x_2^2 - \zeta x_2 + \omega_3, \quad g(x_2) = \eta x_2 + \omega_1,$$

where ω_1, ω_3 and ζ are three constants. Introduction of these two expressions into those of $X(\mathbf{x})$ and $Z(\mathbf{x})$ leads to

$$X(\mathbf{x}) = \omega_1 + \zeta x_1 + \eta x_2 + 2\theta x_1 x_2,$$

$$Z(\mathbf{x}) = \omega_3 - \zeta x_2 - \theta x_2^2.$$

To determine the form of $Y(\mathbf{x})$, it suffices to use the relation $Y(\mathbf{x}) = y(x_1)$, together with

$$\frac{\mathrm{d}y(x_1)}{\mathrm{d}x_1} = X_{,2}(\mathbf{x}) = \eta + 2\theta x_1.$$

Thus,

$$Y(\mathbf{x}) = y(x_1) = X_{,2}(\mathbf{x}) = \omega_2 + \eta x_1 + \theta x_1^2,$$

where ω_2 is a constant. The foregoing expressions for $X(\mathbf{x})$, $Y(\mathbf{x})$ and $Z(\mathbf{x})$, together with their definitions, show the necessity of (3.30c)–(3.30e).

To determine $\alpha(\mathbf{x})$, we return to equation (h′). With the expressions of $\mathbb{W}(\mathbf{x})$ just found out, (h′) implies that

$$\alpha_{,11}(\mathbf{x}) = \alpha_{,22}(\mathbf{x}) = \alpha_{,12}(\mathbf{x}) = 0.$$

This indicates that $\alpha(\mathbf{x})$ necessarily takes the form of the first expression of (3.30a).

Above, we have proved that each of the expressions (3.30a)–(3.30e) is necessary. If they are substituted into (a′), it is easily verified that (a′) holds regardless of function ψ; this gives the sufficiency of (3.30a)–(3.30e).

If (3.30a)–(3.30e) are introduced into (3.27a), we have

$$\mathbb{C}(\mathbf{x}) = \mathbb{W}(\mathbf{x}) + (\mathbf{b}\cdot\mathbf{x} + c)\mathbb{R}, \tag{3.31}$$

where $\mathbb{W}(\mathbf{x})$ is specified by (3.30b)–(3.30e). By means of (3.17) and (2.7), we obtain

$$\mathbb{D}(\mathbf{x}) = \mathbb{V}(\mathbf{x}) + (\mathbf{b}\cdot\mathbf{x} + c)\mathbb{R} \quad \text{or}\; C_{ijmn}(\mathbf{x}) = V_{ijmn}(\mathbf{x}) + (b_k x_k + c)R_{ijmn}. \tag{3.32}$$

Here the tensor $\mathbb{V}(\mathbf{x})$ is related to $\mathbb{W}(\mathbf{x})$ by

$$\mathbb{V}(\mathbf{x}) := \mathbb{R}\mathbb{W}(\mathbf{x})\mathbb{R}. \tag{3.33}$$

Bearing in mind (2.7), it is easy to verify that $\mathbb{V}^T(\mathbf{x}) = -\mathbb{V}(\mathbf{x})$. The components of $\mathbb{V}(\mathbf{x})$ can be determined via (3.33) and (3.30b)–(3.30e):

$$V_{1111}(\mathbf{x}) = V_{2222}(\mathbf{x}) = V_{1212}(\mathbf{x}) = 0, \tag{3.34a}$$

$$V_{2211}(\mathbf{x}) = -V_{1122}(\mathbf{x}) = \omega_1 + \zeta x_1 + \eta x_2 + 2\theta x_1 x_2, \tag{3.34b}$$

$$V_{1211}(\mathbf{x}) = -V_{1112}(\mathbf{x}) = \omega_3 - \zeta x_2 - \theta x_2^2, \tag{3.34c}$$

$$V_{1222}(\mathbf{x}) = -V_{2212}(\mathbf{x}) = \omega_2 + \eta x_1 + \theta x_1^2. \tag{3.34d}$$

Thus we have completed the proof of the following assertion.

Theorem 3.4. Let $\mathscr{B}$ be a simply-connected plane solid which is made of a linearly elastic material M characterized by an elastic tensor field $\mathbb{K}(\mathbf{x})$, and is subjected to prescribed traction forces on its boundary. Consider another linearly elastic material M' which is characterized by an elastic tensor field $\mathbb{K}'(\mathbf{x})$ and is used to substitute for M. Then the stress tensor field of $\mathscr{B}$ is unchanged with the substitution of M' for M if and only if $\mathbb{K}'(\mathbf{x})$ is related to $\mathbb{K}(\mathbf{x})$ as follows:

$$\mathbb{K}'(\mathbf{x}) = \mathbb{K}(\mathbf{x})+\mathbb{V}(\mathbf{x})+(\mathbf{b}\cdot\mathbf{x}+c)\mathbb{R}, \tag{3.35a}$$

with

$$\begin{aligned}\mathbb{V}(\mathbf{x}) = {} & (\omega_1+\zeta x_1+\eta x_2+2\theta x_1 x_2)(\mathbf{e}_2\otimes\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_1-\mathbf{e}_1\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_2)\\ & +(\omega_3-\zeta x_2-\theta x_2^2)[(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\otimes\mathbf{e}_1\otimes\mathbf{e}_1-\mathbf{e}_1\otimes\mathbf{e}_1\otimes(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)]\\ & +(\omega_2+\eta x_1+\theta x_1^2)[(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\otimes\mathbf{e}_2\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_2\otimes(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)].\end{aligned} \tag{3.35b}$$

In comparison with the extended relation (1.2) of Dundurs and Markenscoff (1993), (3.35a) contains an additional term $\mathbb{V}(\mathbf{x})$. This term is a quadratic function of $\mathbf{x}$ and involves six constants. Obviously, when both $\mathbb{K}(\mathbf{x})$ and $\mathbb{K}'(\mathbf{x})$ are supposed to be hyperelastic, $\mathbb{V}(\mathbf{x})$ reduces to the 4th-order zero tensor.

4. CONCLUDING REMARKS

We have answered the question whether there is a more general relation than (1.2) for an elastic tensor field change not to alter the stress state of a plane linearly elastic solid undergoing prescribed traction forces on its boundary. In answering this question, three steps have turned out to be decisive. The first one was a careful reformulation of the original problem of Cherkaev *et al.* (1992) via the orthogonal decompositions of hyperelastic and elastic tensors. The second one was to construct necessary conditions, prior to sufficient conditions, by means of the solutions of some particular traction boundary value problems of plane linear elasticity. The third one was to show that the constructed necessary conditions are also sufficient. As a consequence of the last step, use of some particular traction boundary value problems did not affect the generality of our results.

The conclusion asserting the necessity of (1.2) for the hyperelastic case is important, since it completes the results of Cherkaev *et al.* (1992) and Dundurs and Markenscoff (1993), which have been shown to have a large number of significant applications in the mechanics of composite materials (see, e.g., Thorpe and Jasiuk, 1992; Moran and Gosz, 1994; Zheng and Hwang, 1996, 1997). Our theoretical result regarding the existence of a more general necessary and sufficient relation in the elastic case is new and has practical applications. For example, it can be applied to non-associated elastoplastic materials, where the stresses and strains are incrementally related by a 4th-order tensor presenting the permutation index symmetries of an elastic tensor. Another domain of application is photoelasticity, since the photoelastic tensor has the same algebraic structure as an elastic tensor.

Acknowledgment—The author would like to thank A. Curnier, B. Dacorogna and Q.-S. Zheng for valuable suggestions.

REFERENCES

Backus, G. (1970) A geometrical picture of anisotropic elastic tensors. *Rev. Geophys. Spacephys.* **8,** 633–671.

Chen, T. Y. (1995) Further results on invariant properties of the stress in plane elasticity and its extension to piezoelectricity. *Mech. Res. Comm.* **22,** 251–256.

Cherkaev, A. V., Lurie, K. A. and Milton, G. W. (1992) Invariant properties of the stress in plane elasticity and equivalence classes of composites. *Proceedings of the Royal Society of London A* **438,** 519–529.

Christensen, R. M. (1993) Effective properties of composite materials containing voids. *Proceedings of the Royal Society of London A* **440,** 461–473.

Dundurs, J. and Markenscoff, X. (1993) Invariance of stresses under a change in elastic compliances. *Proceedings of the Royal Society of London A* **443**, 289–300.

Duvaut, G. (1990) *Mécanique des Milieux Continus*. Masson, Paris.

He, Q. C. (1997) A remarkable tensor in plane linear elasticity. *ASME J. Appl. Mech.* **64**, 704–707.

He, Q. C. and Curnier, A. (1995) A more fundamental approach to damaged elastic stress–strain relations. *International Journal of Solids and Structures* **32**, 1433–1457.

Jasiuk, I., Chen, J. and Thorpe, M. F. (1994) Elastic moduli of materials with polygonal and elliptical holes. *Applied Mechanics Review* **47**, S18–S28.

Jun, S. and Jasiuk, I. (1993) Elastic moduli of two-dimensional composites with sliding inclusions—a comparison of effective medium theories. *International Journal of Solids and Structures* **30**, 2501–2523.

Knops, R. J. and Payne, L. E. (1971) *Uniqueness Theorems in Linear Elasticity*. Springer, Berlin.

Milton, G. W. and Movchan, A. B. (1995) A correspondence between plane elasticity and the two-dimensional real and complex dielectric equations in anisotropic media. *Proceedings of the Royal Society of London A* **451**, 293–317.

Moran, B. and Gosz, M. (1994) Stress invariance in plane anisotropic elasticity. *Modelling Simul. Mater. Eng.* **2**, 677–688.

Muskhelishvili, N. I. (1963) *Some Basic Problems of the Mathematical Theory of Elasticity*. P. Noordhoff, Groningen.

Ostoja-Starzewski, M. and Jasiuk, I. (1995) Stress invariance in planar Cosserat elasticity. *Proceedings of the Royal Society of London A* **451**, 453–470.

Spencer, A. J. M. (1970) A note on the decomposition of tensors into traceless symmetric tensors. *International Journal of Engineering Science* **8**, 475–481.

Thorpe, M. F. and Jasiuk, I. (1992) New results in the theory of elasticity for two-dimensional composites. *Proceedings of the Royal Society of London A* **438**, 531–544.

Zheng, Q. S. and Hwang, K. C. (1996) Reduced dependence of defect compliance on matrix and inclusion elastic properties in two-dimensional elasticity. *Proceedings of the Royal Society of London A* **452**, 2493–2507.

Zheng, Q. S. and Hwang, K. C. (1997) Two-dimensional elastic compliances of materials with holes and microcracks. *Proceedings of the Royal Society of London A* **453**, 353–364.